Twice-Ramanujan Sparsifiers

June 24, 2022.

Aim: · Compute κ-sparsifier H of graph G.

$\Rightarrow$ Laplacian

$$x^T L_G x \le x^T L_H x \le \kappa \cdot x^T L_G x$$

s.t. number of edges is less in H.

W

$D = \mathrm{diag}$

$D_{ii} = \sum_{a:i} w_i$

* Thm: $\forall\ G = (V, E, w)$: Undirected weighted graph
$\exists\ H = (V, E, \tilde{w})$: $\lceil d(n-1) \rceil$ edges

$$x^T L_G x \le x^T L_H x \le \underbrace{\left(\frac{d+1+2\sqrt{d}}{d+1-2\sqrt{d}}\right)}_{\kappa} \cdot x^T L_G x.$$

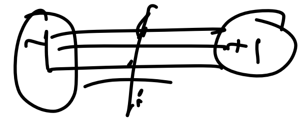
complete $\Rightarrow$ $d(n-1)$

Ramanuja sparsifiers.

$$1 + \frac{4\sqrt{d}}{d+1-2\sqrt{d}}$$

$$1 + \frac{4}{\sqrt{d}-2}$$

Preliminaries



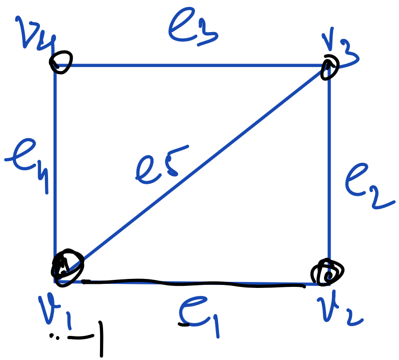
- $L_G = D - W$

- $x^T L_G x = \sum_{(u,v)\in E} w_{u,v}(x_u - x_v)^2$

$$x = \begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \\ -1 \end{pmatrix}$$

- Incidence Matrix : $B_{m\times n}$ (m-edges, n-vertices)

$$B(e,v) = \begin{cases} 1 & \text{if } v \text{ is } e\text{'s head} \\ -1 & \text{if } v \text{ is } e\text{'s tail} \\ 0 & \text{otherwise} \end{cases}$$



$$B = \begin{array}{c|cccc} & v_1 & v_2 & v_3 & v_4 \\ \hline e_1 & -1 & 1 & 0 & 0 \\ e_2 & 0 & -1 & 1 & 0 \\ e_3 & 0 & 0 & -1 & 1 \\ e_4 & 1 & 0 & 0 & -1 \\ e_5 & -1 & 0 & 1 & 0 \end{array}$$

$W_{m\times m} = \mathrm{diag}(w_e)$

Note: $x^T L_G x = x^T B^T W B x = \|W^{1/2} B x\|_2^2$

$$= \sum_{(u,v)\in E} w_{uv} (x_u - x_v)^2 \quad , \ \forall x \in \mathbb{R}^n.$$

- G connected $\Leftrightarrow$ $\mathrm{Ker}(L) = \mathrm{Ker}(W^{1/2} B) = \mathrm{span}(\mathbb{1})$

Rank-1 Updates

* Sherman-Morrison): If A is non singular $n \times n$, v is a vector, then

$$\underbrace{(A + vv^T)^{-1}} = A^{-1} - \frac{A^{-1} v v^T A^{-1}}{1 + v^T A^{-1} v} :$$

$$\Downarrow$$

* (Matrix Determinant Lemma):

$$\underline{\det(A + vv^T)} = \underline{\det(A)}\,(\underline{1 + v^T A^{-1} v})$$

$$A(I + A^{-1} v v^T)$$

Proof Sketch: Factor $A + vv^T = A\left(I + A^{-1}vv^T\right)$

Now consider $(I + uv^T)$:

$$(I + uv^T)u = (1 + v^Tu)u$$

eigenvecs $=$ u ; $\operatorname{span}(u^\perp)$

eigenvals $=$ $(1 + u^Tv)$; 1

$(I + vv^T)^{-1}$ $\Rightarrow$ same eigenvecs: v, $\operatorname{span}(v^\perp)$

eigenvals: $\dfrac{1}{(1+\|v\|^2)}$, 1

$\therefore$ Any matrix of form $(I + g\,vv^T)$ must have all 1's eigenval except one $1 + g\|v\|^2$.

$$\because \quad (I + vv^T)(I + vv^T)^{-1} = I$$

$$\Downarrow$$

$$\Rightarrow \quad (I + vv^T)(I + g\,vv^T) = I$$

$$\Rightarrow \quad I + vv^T + g\,vv^T + v\,g\|v\|^2 v^T = I$$

$$= \quad I + v\,(1 + g + g\|v\|^2)\,v^T = I$$

$$\Rightarrow \quad (1 + g + g\|v\|^2) = 0$$

$$\therefore \quad g = \frac{-1}{1 + \|v\|^2}$$

$$\therefore \quad (I + vv^T)^{-1} = I + v\underbrace{\left(\frac{-1}{1 + v^Tv}\right)}_{\Rightarrow g} v^T .$$

∎

Main Result

* Thm: Suppose $d > 1$, $v_1, \dots, v_m \in \mathbb{R}^n$ s.t.

$$\sum_i v_i v_i^T = I_n . \qquad m > n$$

Then $\exists$ scalar $\underset{i \in [m]}{s_i} \geq 0$ with $|\{i : s_i \neq 0\}| \leq dn$ with

$$I_n \preceq \sum_i s_i v_i v_i^T \preceq \left(\frac{d+1+2\sqrt{d}}{d+1-2\sqrt{d}} \right) I_n$$

$\boxed{\text{Theorem} \Rightarrow \text{Sparsification}}$

- Proof of Thm $\Rightarrow$ Sparsification :

Wlog G is connected. $L_G = B^T W B$. Fix $d > 1$.
Restrict attention to

$$Im(L_G) \equiv \mathbb{R}^{n-1}.$$

Apply Thm to columns $\{v_i\}_{i \le m}$ of

$$V_{n \times m} = (L_G^+)^{1/2} B^T W^{1/2}.$$

Note: $\sum_i v_i v_i^T = VV^T = (L_G^+)^{1/2} \underbrace{B^T W B}_{L_G} (L_G^+)^{1/2}$

$$= (L_G^+)^{1/2} (L_G) (L_G^+)^{1/2} = I_{im(L_G)}$$

- Now, we construct sparsifier graph from theorem.

Let $S_{m\times m} = \mathrm{diag}(s_i)$.

$$L_H = B^T W^{\frac{1}{2}} S W^{\frac{1}{2}} B \Rightarrow \text{subgraph of } G \text{ with weights } (\tilde{w}_i = w_i s_i)$$

Also:

$$I_{\mathrm{im}(L_G)} \preceq \sum s_i v_i v_i^T = VSV^T \preceq K \cdot I_{\mathrm{im}(L_G)}$$

$\therefore$ we get

$$1 \le \frac{y^T V S V^T y}{y^T y} \le K$$

$$x^T L_G x \le x^T L_H x \le K \cdot x^T L_G x$$

$$\Rightarrow 1 \le \frac{y^T (L_G^+)^{\frac{1}{2}} L_H (L_G^+)^{\frac{1}{2}} y}{y^T y} \le K \quad (\forall y \in \mathrm{Im}(L_G))$$

$$(\because V = (L_G^+)^{\frac{1}{2}} B^T W^{\frac{1}{2}})$$

$$\Rightarrow 1 \le \frac{x^T \cancel{L_G^{\frac{1}{2}}} (\cancel{L_G^+})^{\frac{1}{2}} L_H (\cancel{L_G^+})^{\frac{1}{2}} \cancel{L_G^{\frac{1}{2}}} x}{x^T L_G^{\frac{1}{2}} L_G^{\frac{1}{2}} x} \quad (\forall x \perp \mathbb{1})$$

$$Lx = y$$

$$\Rightarrow \boxed{1 \le \frac{x^T L_H x}{x^T L_G x} \le k} \qquad (\forall x \perp \mathbb{1})$$

$\therefore$ Theorem $\Rightarrow$ Sparsification

(Recall)

* **Thm:** Suppose $d > 1$, $v_1, \dots, v_m \in \mathbb{R}^n$ s.t.

$$\sum_i v_i v_i^T = I_{n \times n}.$$

Then $\exists$ scalar $s_i \ge 0$ with $|\{i : s_i \ne 0\}| \le dn$ with

$$I_n \preceq \sum_i s_i v_i v_i^T \preceq \left(\frac{d+1+2\sqrt{d}}{d+1-2\sqrt{d}}\right) I_n$$

Proof Intuition

- Eigenvalues of $\underline{A}$ $\underline{\text{interlace}}$ eigenvalues of $\underline{A+vv^T}$.

⇒ We can determine the new eigenvalues:

$$\boxed{p_{A+vv^T}(x)} = p_A(x)\left(1-\sum_j \frac{\langle v,u_j\rangle^2}{x-\lambda_j}\right)$$

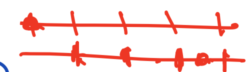
$\det(xI - A - vv^T)$

$\det(A+vv^T) = \det(A)\cdot(I+A^{-1}vv^T)$

The $p_{A+vv^T}(x)$ has 2 kinds of zeros λ:

1. Those for which $p_A(\lambda)=0$, i.e. $v \perp u_j$
2. Those for which $p_A(\lambda)\neq 0$ & $p_{A+vv^T}(\lambda)=0$

⇒ $$f(\lambda) := \left(1-\frac{1}{m}\sum_j \frac{\langle v,u_j\rangle^2}{x-\lambda_j}\right) = 0$$



⇒ These eig have "moved", & interlace old eigenvals.

- Now suppose we add random vec. : $A + v_i v_i^T$

$$\mathbb{E}_v \langle v, u_j \rangle^2 = \frac{1}{m} \sum_i \langle v_i, u_j \rangle^2 = \frac{1}{m} u_j^T \left(\sum_i v_i v_i^T \right) u_j$$

$\sum v_i v_i^T = I$

$$= \frac{\|u_j\|^2}{m} = \frac{1}{m}.$$

$\therefore$ Adding this "average" vec. v_i gives:

$$p_{A+v_iv_i^T}(x) = p_A(x) \left(1 - \left(\sum_i \frac{1/m}{x - \lambda_i} \right) \right) = p_A(x) - \frac{1}{m} p_A'(x).$$

$p_A(x) = \prod_i (x - \lambda_i)$

$$\because \quad p_A' = \sum_j \prod_{i \neq j} (x - \lambda_i)$$

Start with $A = 0 \Rightarrow p_A(x) = x^m$

$$p_{A+vv^T}(x) = \left(p_A(x) - \frac{1}{m} p_A'(x) \right) - \frac{1}{m} p'_{A+vv^T}(x)$$

$\vdots$

$\cdots \Rightarrow$ Laguerre Polynomial

"Well-known" for Laguerre that after dn iterations:

$$\boxed{\frac{\text{Largest zero}}{\text{Smallest zero}}} = \boxed{\frac{d+1+2\sqrt{d}}{d+1-2\sqrt{d}}}.$$

(Recall)

* Thm: Suppose $d > 1$, $v_1, \dots, v_m \in \mathbb{R}^n$ s.t.

$$\sum_i v_i v_i^T = I_{n\times n}.$$

Then $\exists$ scalar $s_i \geq 0$ with $|\{i : s_i \neq 0\}| \leq dn$ with

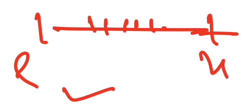
$$I_n \preceq \sum_i s_i v_i v_i^T \preceq \left(\frac{d+1+2\sqrt{d}}{d+1-2\sqrt{d}}\right) I_n$$

Proof by Barrier Functions

* Def: $u, l \in \mathbb{R}$, A symm $\lambda_1, \dots, \lambda_n$

$$\Phi^u(A) := \mathrm{tr}(uI - A)^{-1} = \sum_i \frac{1}{u - \lambda_i}$$ (upper potential)

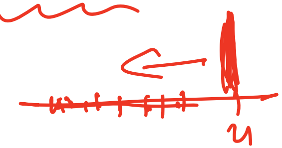
$$\Phi^l(A) := \mathrm{tr}(A - lI)^{-1} = \sum_i \frac{1}{\lambda_i - l}$$ (lower potential)



* Rmk: As long as $A \prec uI$, $A \succ lI$, potential measures how "far" eig(A) are from barriers u & l.

* <u>Intuition</u>: Suppose $\Phi^u(A) \le 1$, & $A \preceq uI$. Then,

$$\Phi^u(A) = \sum_i \frac{1}{u-\lambda_i} \le 1$$



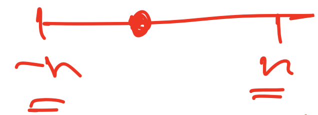
$\Rightarrow$ At most 1 eig at dist 1 from u.

At most 2 eig at dist 2 from u.

$\vdots$

· $\Phi^u(A)$ measures total "repulsion" of eigenvalues of A from potential u.

* Steps to prove the theorem:



· Iteratively build $\sum_i s_i v_i v_i^T$. close to I

· Start from $A^{(0)} = 0$

$A^{(1)} = A^{(0)} + t v_i v_i^T$

$\vdots$

· Consider the consts: $u_0,\ l_0,\ \delta_u,\ \delta_L,\ \varepsilon_u,\ \varepsilon_L$

(a) Initial Barriers: $u = u_0$, $l = l_0$, & potentials:

$$\Phi^{u_0}(A^{(0)}) = \varepsilon_U$$

$$\Phi^{l_0}(A^{(1)}) = \varepsilon_L$$

$$\sum \frac{1}{u_0 - \lambda_i} = \varepsilon_U$$

$\Rightarrow$

$$\sum \frac{1}{u - \lambda_i} = \sum \frac{1}{u + \delta_u - \lambda_i}$$

$A + t v v^T$

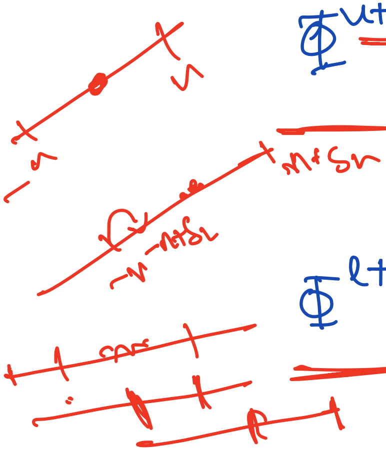
(b) Each matrix obtained by:

$$A^{(q+1)} = A^{(q)} + t\,v v^T, \qquad v \in \{v_1, \dots, v_m\}, \quad t \geq 0.$$

(c) Increment Barriers: $u \to u + \delta_U$ & $\ell \to \ell + \delta_L$:

$\Phi^U(A) = \varepsilon_U$

$\Phi^{U+\delta_U}(A) < \Phi^U(A)$



$$\Phi^{u+\delta_U}\big(\overset{A^q + t v v^T}{A^{(q+1)}}\big) \leq \Phi^U(A^{(q)}) \leq \delta_U,$$

where $u = u_0 + q\,\delta_U$

$$\Phi^{\ell+\delta_L}(A^{(q+1)}) \leq \Phi^{\ell}(A^{(q)}) \leq \delta_L$$

where $\ell = \ell_0 + q\,\delta_L$

(d) No eig jumps across barrier:

$$\lambda_{max}(A^{(q)}) \leq u_0 + q\,\delta_U \quad \& \quad \lambda_{min}(A^{(q)}) \geq \ell_0 + q\,\delta_L.$$

Combining (a), (b), (c), (d), we get: for $Q = dn$

$$\frac{\lambda_{max}(A^{(Q)})}{\lambda_{min}(A^{(Q)})} \le \frac{u_0 + dn\,\delta_U}{l_0 + dn\,\delta_L} = \frac{d+1+2\sqrt{d}}{d+1-2\sqrt{d}}$$

by construction: $A^{(Q)}$ is weighted sum of at most dn vectors.

$+ v_i v_i^T$

* Lemma 1: (Upper Barrier Shift)

Shifting u to $u+\delta u$, we observe

$$\Phi^{u}(A) > \Phi^{u+\delta u}(A).$$

$\therefore$ Can add $(A + tvv^T)$ s.t.

$$\Phi^{u}(A) > \Phi^{u+\delta u}(A+tvv^T).$$

$\therefore$ $t = ?$

Suppose $\lambda_{max}(A) \le u$, v is any vector. If

$$\frac{1}{t} \geq v\left(\frac{((u+\delta u)I-A)^{-2}}{\Phi^{u}(A)-\Phi^{u+\delta u}(A)} + ((u+\delta u)I-A)^{-1}\right)v^T =: U_A(v)$$

$$\Phi^{u+\delta u}(A+tvv^T) \le \Phi^{u}(A) \ \vee \ \lambda_{max}(A+tvv^T) \le u+\delta u.$$

* **Proof**: Let $u' = u + \delta u$.

$$\Phi^{u+\delta u}(A + t\, vv^T) = \mathrm{tr}\,(u'I - A - t\,vv^T)^{-1}$$

$$= \mathrm{tr}\left((u'I - A)^{-1} + \frac{t\,(u'I-A)^{-1} vv^T (u'I-A)^{-1}}{1 - t\,v^T (u'I-A)^{-1} v} \right)$$

$$= \mathrm{tr}\,(u'I - A)^{-1} + t \cdot \frac{v^T (u'I-A)^{-2} v}{1 - t\,v^T (u'I-A)^{-1} v}$$

$$= \Phi^{u}(A) - \left(\Phi^{u}(A) + \Phi^{u'}(A)\right) + \underbrace{\frac{v^T (u'I-A)^{-2} v}{1/t - v^T (u'I-A)^{-1} v}}$$

$\because$

$$U_A(v) > v^T (u'I-A) v \quad \& \quad 1/t \geq U_A(v)$$

$\therefore$ <u>Last term</u> is finite

$\Rightarrow$ By choice of $1/t$

$$\underbrace{\frac{v^T(U'I-A)^{-2}v}{1/t - v^T(U'I-A)^{-1}v}}_{\text{Last term}}$$

$$\frac{1}{t} \geq v\left(\frac{((u+\delta u)I-A)^{-2}}{\Phi^{u}(A)-\Phi^{u+\delta u}(A)} + ((u+\delta u)I-A)^{-1}\right)v^T,$$ we get

$$\Phi^{u+\delta u}(A) \leq \Phi^{u}(A).$$

Moreover since $\underbrace{\text{last term}}$ is finite,

$$\lambda_{max}(A + t v v^T) < u + \delta u.$$

because if not, then for some +ve $t' < t$ $\lambda_{max}(A + t' v v^T) = u + \delta u$. But at such t' $\Phi^{u+\delta u}(A + t' v v^T)$ would be ∞ $\Rightarrow$ contradiction

$\blacksquare$

* Lemma 2: (lower Barrier)

Suppose $\lambda_{min}(A) > \ell$, $\Phi_\ell(A) \le 1/\delta_L$, v any vec.

If,

$$0 < \frac{1}{t} \le v^T\left(\frac{(A - (\ell+\delta_L)I)^{-2}}{\Phi^{\ell+\delta_L}(A) - \Phi^{\ell}(A)} - (A - (\ell+\delta_L)I)^{-1} \right) v =: L_A(v)$$

Then,

$U_A(v) \le \frac{1}{t} \le L_A(v)$

$$\Phi_{\ell+\delta_L}(A + tvv^T) \le \Phi_\ell(A) \quad \text{and}$$

$$\lambda_{min}(A + tvv^T) > \ell + \delta_L$$

Same proof.

* Lemma 3 : (Both Barriers Simultaneously)

If $\lambda_{max}(A) < u$, $\lambda_{min}(A) > l$, $\Phi^u(A) \le \varepsilon_U$, $\Phi_l(A) \le \varepsilon_L$,

and $\varepsilon_U, \varepsilon_L, \delta_U$ & δ_L satisfy :

$$0 \le \frac{1}{\delta_U} + \varepsilon_U \le \frac{1}{\delta_L} - \varepsilon_L .$$ Then $\exists\, i$, $t > 0$ s.t.

$L_A(v_i) \ge \frac{1}{t} \ge U_A(v_i)$, $\lambda_{max}(A + t v_i v_i^T) < u + \delta_u$,

$\lambda_{min}(A + t v_i v_i^T) > l + \delta_L$.

* **Proof** : We will show :

$$\sum_i L_A(v_i) \geq \sum_i U_A(v_i)$$

$$\Downarrow$$

$\exists$ vector v_i s.t. $L_A(v_i) \geq U_A(v_i)$

$$\sum U_A(v_i) = \sum_i v_i \left(\frac{((u+\delta u)I - A)^{-2}}{\Phi^u(A) - \Phi^{u+\delta u}(A)} + ((u+\delta u)I - A)^{-1} \right) v_i^T$$

$$= \frac{((u+\delta u)I - A)^{-2}}{\Phi^u(A) - \Phi^{u+\delta u}(A)} \cdot \underbrace{\left(\sum_i v_i v_i^T\right)}_{=I} + ((u+\delta u)I - A)^{-1} \cdot \underbrace{\left(\sum_i v_i v_i^T\right)}_{=I}$$

$$= \operatorname{tr}\left(\frac{((u+\delta u)I - A)^{-2}}{\Phi^u(A) - \Phi^{u+\delta u}(A)} \right) + \operatorname{tr}\left(((u+\delta u)I - A)^{-1} \right)$$

$$= \frac{\sum_i (u+\delta u - \lambda_i)^{-2}}{\sum_i (u-\lambda_i)^{-1} - \sum_i (u+\delta u - \lambda_i)^{-1}} + \Phi^{u+\delta u}(A)$$

$$= \frac{\sum_i (u+\delta u - \lambda_i)^{-2}}{\delta u \left(\sum_i (u-\lambda_i)^{-1} \cdot (u+\delta u - \lambda_i)^{-1} \right)} + \Phi^{u+\delta u}(A)$$

$\because \quad \sum (u-\lambda_i)^{-1} (u+\delta u-\lambda_i)^{-1} \geq \sum_i (u+\delta u-\lambda_i)^{-2}$

$\Rightarrow \sum_i U_A(v_i) \leq \frac{1}{\delta_L} + \Phi^{u+\delta u}(A) \leq \frac{1}{\delta u} + \varepsilon_u$

Similarly,

$\sum_i L_A(v_i) \geq \frac{1}{\delta_L} - \sum_i (\lambda_i - l)^{-1} = \frac{1}{\delta_L} - \varepsilon_L$

Putting these together

$\sum_i U_A(v_i) \leq \frac{1}{\delta_U} + \varepsilon_U \leq \frac{1}{\delta_L} - \varepsilon_L \leq \sum_i L_A(v_i)$ $\blacksquare$

Proof of Theorem

We need to set $\varepsilon_u, \varepsilon_L, \delta_u, \delta_L$ s.t. Lemma 3 is satisfied.

Recall: constructed $A^{(q+1)} = A^{(q)} + t v_i v_i^T$, $t \geq 0$

$\rightarrow$ $L_{A^{(q)}}(v_i) \geq \frac{1}{t} \geq U_{A^{(q)}}(v_i)$

Take $\delta_L = 1$, $\varepsilon_L = \frac{1}{\sqrt{d}}$, $l_0 = -n/\varepsilon_L$

$\delta_U = \frac{\sqrt{d}+1}{\sqrt{d}-1}$, $\varepsilon_U = \frac{\sqrt{d}-1}{d+\sqrt{d}}$, $u_0 = n/\varepsilon_U$

* Check that

$$\frac{1}{\delta_U} + \varepsilon_U = \frac{1}{\delta_L} - \varepsilon_L$$

$-n \longmapsto n$

ε_L ε_U

$u + \delta_U$ $l + \delta_L$

Initial potentials are: $\Phi^{\frac{n}{\varepsilon_U}}(0) = \varepsilon_U$

$$\Phi^{\frac{n}{\varepsilon_L}}(0) = \varepsilon_L$$

$$\frac{\lambda_{max}\left(A^{(dn)}\right)}{\lambda_{min}\left(A^{(dn)}\right)} \leq \frac{n/\varepsilon_U + dn\,\delta_U}{-n/\varepsilon_L + dn\,\delta_L}$$

$$= \frac{\dfrac{d+\sqrt{d}}{\sqrt{d}-1} + \dfrac{d\sqrt{d}+1}{\sqrt{d}-1}}{d-\sqrt{d}}$$

$$= \frac{d + 2\sqrt{d} + 1}{d - 2\sqrt{d} + 1}.$$

$\blacksquare$

The Algorithm:

- Compute vectors $v_i \Rightarrow \quad O(n^2 m)$
- Compute $((u+\delta u)I - A)^{-1}, ((u+\delta u)I - A)^{-2}$ in each iter

$$= O(n^3)$$

- Compute $U_A(v_i)$ & $L_A(v_i) \quad = O(n^2 m)$
- Run for (dn) iterations $\Rightarrow$

$$O(d n^3 m).$$